Analytic Fredholm Theory

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The purpose of this note is to prove a version of analytic Fredholm theory, and examine a special case.

Theorem 1.1 (Analytic Fredholm Theory). *Let Ω be a connected open subset of $\mathbf{C}$ and suppose $T(\lambda)$ is an analytic family of Fredholm operators on a Hilbert space H. Then either*

(i) $T(\lambda)$ is not invertible for any $\lambda \in \mathbf{C}$, or

(ii) There exists a discrete set $S \subseteq \Omega$ such that $T(\lambda)$ is invertible for all $\lambda \notin S$ and furthermore $T^{-1}(\lambda)$ extends to a meromorphic function on all of Ω. Furthermore, every operator appearing as a coefficient of a term of negative order is finite rank.

By *analytic*, we mean that for every $\lambda_0 \in \Omega$, T is given by a power series

$$T(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n T_n$$

converging in the operator norm, where $T_n : H \to H$ are bounded. Similarly, we say that T is *meromorphic* if around every $\lambda_0 \in \Omega$ there is a Laurent series

$$T(\lambda) = \sum_{n=-N}^{\infty} (\lambda - \lambda_0)^n T_n$$

converging in a punctured neighbourhood of λ_0.

We will also prove:

Proposition 1.2. *Suppose $T : H \to H$ is a self-adjoint, non-negative bounded operator. Suppose that $T - \lambda$ is Fredholm for all $\lambda \in \Omega$, where $\Omega \subset \mathbf{C}$ is a connected open set. Then $(T - \lambda)^{-1}$ has a meromorphic extension to a family with only simple poles, and the residue is -1 times the projection onto the kernel of $T - \lambda_0$, where λ_0 is a pole.*

Proof of Analytic Fredholm Theory. We divide the proof into several steps.

1. Show that if $T^{-1}(\lambda)$ exists as an operator at a point, it is analytic in a neigbhourhood of that point.

2. Show that if $T^{-1}(\lambda)$ has a meromorphic extension near a point, it is analytic in a punctured neigbhourhood of that point (this step actually applies to any meromorphic family).

3. Show that if $T^{-1}(\lambda)$ has a meromorphic extension to a connected open set U on which $T(\lambda')$ is invertible for at least one $\lambda' \in U$, then the points where $T^{-1}(\lambda)$ are analytic are in fact points where the inverse actually exists, and the points where it fails to be analytic form a discrete set. In particular, this is true if U contains λ_0.

4. Show that $T^{-1}(\lambda)$ has a meromorphic extension to the union Ω' of all connected open sets U containing λ_0 on which $T^{-1}(\lambda)$ has an extension, and that Ω' is open, connected, and non-empty.

5. Show that $\Omega' = \Omega$. This last step is the hardest.

Step 3 shows that the points at which $T^{-1}(\lambda)$ fails to exist in Ω' are discrete, and Step 5 will show that $\Omega' = \Omega$ and complete the proof of existence. We will address the finite-rank of the negative-order coefficients later.

Step 1. Fix $\lambda_1 \in \Omega$ for which $T^{-1}(\lambda)$ exists, and write

$$T(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_1)^n T_n.$$

The Cauchy-Hadamard theorem applies, and in particular there exists A, B such that $\|T_n\| \leq AB^n$. Since $T(\lambda_1) = T_0$ is invertible, we may recursively define operators S_n by $S_0 = T_0^{-1}$ and

$$S_n = -(S_{n-1}T_1 + \cdots S_0 T_n)T_0^{-1}.$$

Define $b_0 = \|S_0\|$ and

$$b_n = \|S_0\|(b_{n-1}AB^1 + \cdots b_0 AB^n).$$

It is clear that we have the bound $b_n \leq CD^n$ for some C, D, and that $\|S_n\| \leq b_n \leq CD^n$. In particular, the series

$$S(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_1) S_n$$

converges in a small neighbourhood and by construction $S(\lambda)T(\lambda) = 1$ wherever both are defined. Since $T^{-1}(\lambda)$ exists for λ near λ_1 ($GL(H)$ is open), $S(\lambda) = T^{-1}(\lambda)$ near λ_1. In particular, $T^{-1}(\lambda)$ is analytic near λ_1.

Step 2. Suppose that $T^{-1}(\lambda)$ has a meromorphic extension near a point λ_1. Then

$$T^{-1}(\lambda) = \sum_{n=-N}^{\infty} (\lambda - \lambda_1)^n T_n,$$

and for λ_2 near λ_1,

$$T^{-1}(\lambda) = \sum_{n=-N}^{\infty} (\lambda - \lambda_2 + (\lambda_2 - \lambda_1))^n T_n.$$

If $N \leq 0$ then we may expand $(\lambda - \lambda_2 + (\lambda_2 - \lambda_1))^n$ for $n \geq 0$ using the binomial theorem and regroup terms (since the sum converges absolutely by Cauchy-Hadamard) to see that $T^{-1}(\lambda)$ is analytic near λ_2. If $N > 0$, then we do the same if $n \geq 0$, and if $n < 0$ (and λ close enough to λ_2, we may expand $(\lambda - \lambda_2 + (\lambda_2 - \lambda_1))^n$ as a geometric series and rearrange to see that $T^{-1}(\lambda)$ is analytic near λ_2. In particular, $T^{-1}(\lambda)$ is analytic (and in particular meromorphic) near λ_2. This yields two results: the first is that the set of points for which $T^{-1}(\lambda)$ is meromorphic is open. The second is that inside the set for which $T^{-1}(\lambda)$ has a meromorphic extension, the set points for which $T^{-1}(\lambda)$ is not analytic is discrete.

Step 3. From Step 2 we know that $T^{-1}(\lambda)$ has an analytic extension to a set $S \subseteq U$ with discrete complement. Since from Step 1, the identities

$$T^{-1}(\lambda)T(\lambda) = 1 = T(\lambda)T^{-1}(\lambda)$$

hold on at least a small open subset $\lambda' \in V \subset U$, and $T^{-1}(\lambda)$ makes sense as an analytic function on the connected open set $\Omega' \setminus \{S\} \subseteq \Omega'$ it follows that the identity persist to all of $U \setminus \{S\}$. Indeed, for instance,

$$\langle u(T^{-1}(\lambda)T(\lambda) - 1)v\rangle$$

(for $u, v \in H$) is a $\mathbf{C}$-valued analytic function which is 0 on an open subset, and thus 0 everywhere. Thus not only does the meromorphic extension of $T^{-1}(\lambda)$ fail to be analytic except on an isolated number of points, but $T^{-1}(\lambda)$ is actually the inverse if $T^{-1}(\lambda)$ is analytic and fails to exist at the same points that it fails to be analytic.

Step 4. Let $\{U_\alpha\}$ be the collection of all connected open sets containing λ_0 for which $T^{-1}(\lambda)$ has a meromorphic extension to U_α. Certainly $\Omega' = \bigcup_\alpha U_\alpha$ is open. It is non-empty since by Step 1 is contains a neighbourhood of λ_0. It is connected since every point in Ω' can be connected via a continuous path to λ_0. It remains to show that $T^{-1}(\lambda)$ has a meromorphic extension to Ω'. It suffices to show that if $U_\alpha, U_{\alpha'}$ are open sets as above with non-empty intersection, then the extensions $T^{-1}(\lambda)$ agree on $U_\alpha \cap U_{\alpha'}$. Indeed, by Step 3, $T^{-1}(\lambda)$ is the actual inverse of $T(\lambda)$ at all but discretely many points in $U_\alpha \cap U_{\alpha'}$, and so must agree. Thus the meromorphic extensions of $T^{-1}(\lambda)$ must agree everywhere on $U_\alpha \cup U_{\alpha'}$.

Step 5. Let $\lambda_1 \in \partial\Omega'$.[1]. We need only show that $\lambda_1 \in \Omega'$, since then Ω' is open, closed, and non-empty, and thus all of Ω. Without loss of generality we may take $\lambda_1 = 0$. We have to show that $T^{-1}(\lambda)$ extends to a meromorphic function around $\lambda = 0$. $T(\lambda)$ is a continuous family of Fredholm operators which is invertible at some point λ_0. In particular, the index of $T(\lambda)$ is 0 everywhere. Write $T = T(0)$. Set $V = \ker T^\perp$ and $W = \ker T$, and

[1] By this we mean the boundary with respect to the subspace topology on Ω, i.e. $\partial\Omega' \cap \Omega$, where here ∂ is intepreted as the boundary as a subset of $\mathbf{C}$

set $V' = \operatorname{im} T$ and $W' = \operatorname{im} T^{\perp}$. Since T is Fredholm of index 0, V, W, V', W' are all closed, $V + W = V' + W' = H$ and $T : V \to V'$ is an isomorphism. We denote by Π_X the projection onto the subspace $X \subseteq H$. Notice that if U is any neighbourhoodof 0, then $U \cap \Omega' \neq \emptyset$, and since $T(\lambda)$ is invertible at all but a discrete number of points in Ω' (Step 3), it is invertible at at least one point in $U \cap \Omega' \subseteq U$.

We divide the rest of Step 5 substeps (i) and (ii).

Substep i) In this substep we find a locally invertible analytic family of operators $R(\lambda)$ such that $R(\lambda)T(\lambda)$ looks like the matrix:

$$\begin{pmatrix} A(\lambda) & 0 \\ B(\lambda) & 1 \end{pmatrix}, \tag{1}$$

where the first row and column represent W and the second row and column represent V. The point of this is that we can actually write down a nice inverse for this matrix, at least formally.

Write

$$T(\lambda) = T(\lambda)\Pi_V + T(\lambda)\Pi_W.$$

Then both summands are analytic and we have

$$T(\lambda) = \sum_{n=0}^{\infty} \lambda^n T_n \Pi_V + \sum_{n=0}^{\infty} \lambda^n T_n \Pi_W.$$

By assumption $T(0)\Pi_V = T\Pi_V$ is invertible as an operator from V onto V'. Thus the analytic family $V \to V'$ given by

$$T(\lambda)\Pi_V = \sum_{n=0}^{\infty} \lambda^n T_n \Pi_V$$

is invertible for small λ. Call this inverse $Q(\lambda) : V \to V'$ (which is analytic by an argument like in Step 1. We have of course generalized to the case where the domain and codomain are not the same; however they are still isomorphic). We extend $Q(\lambda)$ to an analytic family of operators $H \to H$ by multiplying on the right by $\Pi_{V'}$. The new operator, $Q_{\text{new}}(\lambda) = Q(\lambda)\Pi_{V'}$ we will henceforth call $Q(\lambda)$, too, in order to reduce unnecessary notation.

Since T has index 0, $\dim W = \dim W' < \infty$, and we may pick an isomorphism P of W' onto W. Of course $PT\Pi_V = 0$. We show that we can extend this to small λ, i.e. find an analytic function $P(\lambda)$ with $P(0) = P$ for which $P(\lambda)T(\lambda)\Pi_V = 0$. Define recursively $P_0 = P$ and

$$P_{k+1} = (\Pi_W - (P_{n-1}T_1 + \cdots P_0 T_n))Q_0.$$

Similiar to Step 1, setting

$$P(\lambda) = \sum_{n=0}^{\infty} \lambda^n P_n$$

defines an analytic family. Since $Q_0 T_0 \Pi_V = \Pi_V$ and $\Pi_W \Pi_V = 0$, $P(\lambda)T(\lambda)\Pi_V = 0$.

Now set $R(\lambda) = Q(\lambda) + P(\lambda)$. Since $R(0)$ is invertible, $R(\lambda)$ is locally invertible with analytic inverse (by Step 1, for instance).

Set $A(\lambda) = \Pi_W R(\lambda)T(\lambda)\Pi_W$. Then $A(\lambda)$ can be interpreted as an analytic family of operators from $W \to W$, i.e. $A(\lambda)$ is an analytic family of square matrices. Similarly, set $B(\lambda) = \Pi_V R(\lambda)T(\lambda)\Pi_W$. Observe that

$$\Pi_W R(\lambda)T(\lambda)\Pi_V = \Pi_W(Q(\lambda)T(\lambda) + P(\lambda)T(\lambda))\Pi_V = \Pi_W \Pi_V + 0 = 0,$$

and similarly

$$\Pi_V R(\lambda)T(\lambda)\Pi_V = \Pi_V \Pi_V = \Pi_V.$$

Writing

$$R(\lambda)T(\lambda) = (\Pi_V + \Pi_W)R(\lambda)T(\lambda)(\Pi_V + \Pi_W)$$

arrives at the matrix representation (1).

Substep ii) From the matrix representation, it is easy to see that formally an inverse should be given by the operator corresponding to

$$\begin{pmatrix} A^{-1}(\lambda) & 0 \\ -B(\lambda)A^{-1}(\lambda) & 1 \end{pmatrix}. \tag{2}$$

In the rest of this substep, we show that this is actually a well-defined meromorphic extension of $(R(\lambda)T(\lambda))^{-1}$ and show that this gives a well-defined meromorphic extension of $T(\lambda)^{-1}$.

Since W, W' are fixed finite dimensional spaces, $A^{-1}(\lambda)$ can be written formally in the form $A^{-1}(\lambda) = p(\lambda)^{-1}\operatorname{Adj}(A(\lambda))$ where $p(\lambda) = \det A(\lambda)$, and $\operatorname{Adj}(A(\lambda))$ is the classical adjugate matrix. Since $A(\lambda)$ is certainly analytic, $p(\lambda)$ is a $\mathbf{C}$-valued analytic function, and $\operatorname{Adj}(A(\lambda))$ is analytic. Now $p(\lambda)$ is not identically 0. Indeed, if it were then $A(\lambda)$ would not be invertible anywhere. This means, using (1), that neither is $R(\lambda)T(\lambda)$. Since $R(\lambda)$ is invertible, this would mean $T(\lambda)$ is not invertible anywhere, which contradicts the fact that $T(\lambda)$ is invertible at at least one point in any neighbourhood of 0.

So $p(\lambda)$ is holomorphic and not identically 0, and so $p(\lambda)^{-1}$ is meromorphic, and thus $A^{-1}(\lambda)$ is meromorphic. Moreover, since A is a square matrix, all coefficients of terms of negative order in the Laurent expansion of $A(\lambda)$ are finite-rank operators.

In particular, the operator family

$$S(\lambda) = A^{-1}(\lambda) - B(\lambda)A^{-1}(\lambda) + \Pi_V,$$

corresponding to the inverse matrix written above, is also meromorphic near λ_1 (we remark that some care needs to be taken when interpreting this formula; one needs to extend $A^{-1}(\lambda)$ to a function on all of H by setting it to be 0 on V'; the extended $A^{-1}(\lambda)$ is still meromorphic). We clearly still have that all coefficients of terms of negative order in $S(\lambda)$ are finite rank. $S(\lambda)$ is actually an inverse to $R(\lambda)T(\lambda)$ wherever $p(\lambda) \neq 0$. Thus $S(\lambda)R(\lambda) = T^{-1}(\lambda)$ in the same area. But $S(\lambda)R(\lambda)$ is meromorphic near λ_1, and thus we conclude that $T^{-1}(\lambda)$ has a meromorphic extension. Thus, $T^{-1}(\lambda)$ exists as a meromorphic family on a connected neighbourhood U of λ_1. Since $\lambda_1 \in \partial\Omega'$, $U \cap \Omega'$ is non-empty. Thus, for the same reason as in the proof of Step 4, $T^{-1}(\lambda)$ extends to a meromorphic family on $U \cup \Omega'$. Since U is connected and $U \cup \Omega'$ is non-empty, $U \cup \Omega'$ is a connected open set containing λ_0, and thus $\lambda_1 \in U \cup \Omega' \subseteq \Omega'$, as desired.

We still need to show that coefficients of terms of negative order in the meromorphic extension are finite rank. Suppose without loss of generality that $T^{-1}(\lambda)$ does not exist as $\lambda = 0$. Write

$$T(\lambda) = \sum_{n=0}^{\infty} T_n \lambda^n$$

and

$$T^{-1}(\lambda) = \sum_{n=-N}^{\infty} R_n \lambda^n,$$

for $N \geq 1$. We need to show that each R_{-k}, $k \geq 1$ has finite rank. Since $T(\lambda)$ is Fredholm for all λ, in particular $T(0) = T_0$ is Fredholm, and so has finite-dimensional kernel. Since $T(\lambda)T^{-1}(\lambda) = 1$, the coefficient of λ^{-k} in the product is 0 for $k \geq 1$. This means that the following equations are valid for $1 \leq k \leq N$:

$$0 = \sum_{j=0}^{N-k} T_j R_{-k-j}.$$

Using this we inductively show that R_{-k} has finite rank. The equation for $k = N$ simply reads $T_0 R_{-N} = 0$. Since $\dim \ker T_0 < \infty$, this means that R_{-N} has finite rank. For $k < N$, we may rearrange the equation to get

$$T_0 R_{-k} = -\sum_{j=1}^{N-k} T_j R_{-k-j}.$$

By induction, the right-hand side has finite-rank, and therefore so does the left. We can write

$$\dim \operatorname{im} R_{-k} = \dim(\ker T_0 \cap \operatorname{im} R_{-k}) + \dim(\ker T_0^{\perp} \cap \operatorname{im} R_{-k}).$$

The first term is finite, since $\dim \ker T_0 < \infty$. Since $T_0 R_{-k}$ has finite rank, $T_0|_{\ker T_0^{\perp} \cap \operatorname{im} R_{-k}}$ has finite rank and is injective, and so the second term is finite, too. □

We now turn to the proof of the proposition:

Proof. By the spectral theorem for self-adjoint operators, $T - \lambda$ is invertible (and hence analytic) off of $[0, \infty)$. Since Ω is open, it necessarily intersects points not in this set. Thus analytic Fredholm theory applies.

Fix $\lambda_0 \in [0, \infty) \cap \Omega$ for which T is not invertible, and set $S = T - \lambda_0$. Since S is self-adjoint and Fredholm, it is invertible as a map $: \operatorname{im} S = \ker S^{\perp} \to \operatorname{im} S$, and $S \equiv 0$ as a map $: \ker S \to \ker S = \operatorname{im} S^{\perp}$. We can thus picture $S - \mu$ as the matrix

$$\begin{pmatrix} -\mu & 0 \\ 0 & S - \mu \end{pmatrix},$$

where the first row and column represent $\ker S$ and the second row and column represent $\operatorname{im} S$. In particular, $S-\mu$ is invertible for small $\mu \neq 0$. Notice that S is invertible on $\operatorname{im} S$. This means that for $v \in \operatorname{im} S$ and $\mu \neq 0$ small,

$$\|(S-\mu)^{-1}v\| \leq (\|S|_{\operatorname{im} S}^{-1}\|^{-1} - \mu)^{-1}\|v\|, \tag{3}$$

which is uniformly bounded as $\mu \to 0$. Write $H \ni u = v + w$, where $v \in \operatorname{im} S$, and $w \in \ker S$. Then, for $\mu \neq 0$ small,

$$\|(S-|\mu|)^{-1}v\|^2 + \|\mu^{-1}w\|^2. \tag{4}$$

For arbitrary $\|u\| = 1$, we can use (3) to bound (4) above by

$$\left(\|S|_{\operatorname{im} S}^{-1}\|^{-1} - |\mu|\right)^{-2} + \mu^{-2} \lesssim \mu^{-2}, \tag{5}$$

We conclude that $\|(S-\mu)^{-1}\| \lesssim \mu^{-1}$ as $\mu \to 0$, so $\|(T-\lambda)^{-1}\| \lesssim |\lambda - \lambda_0|^{-1}$ as $\lambda \to \lambda_0$.

But it is also clear that $\|(T-\lambda)^{-1}\|$ blows up like $|\lambda - \lambda_0|^{-m}$, where m is the order of the pole at λ_0. Indeed, we may write

$$\|(T-\lambda)^{-1}\| = |\lambda - \lambda_0|^{-m}\|R_{-m} + (\lambda - \lambda_0)R_{-n+1} + \cdots\|,$$

and the the series in the second factor converges to a continuous function of λ for λ near λ_0 (by Cauchy-Hadamard, for instance). Thus the second factor converges to $\|R_{-n}\|$, which is nonzero, and so we have the desired blow up.

It follows that $m = 1$, and so the pole is simple.

Next, write

$$(T-\lambda)^{-1} = \sum_{n=-1}^{\infty} (\lambda - \lambda_0)^n R_n$$

around a pole λ_0. We have that

$$(T-\lambda)^{-1}(T-\lambda) = (T-\lambda)(T-\lambda)^{-1} = 1$$

and

$$T - \lambda = (T - \lambda_0) - (\lambda - \lambda_0)$$

are both valid around λ_0 (but of course not at it). Expand the first identity out as a series, and looking at the $n = -1, 0$ terms yields $R_{-1}(T-\lambda_0) = (T-\lambda_0)R_{-1} = 0$ and $R_0(T-\lambda_0) = (T-\lambda_0)R_0 = 1 + R_{-1}$. The first of the two implies that R_{-1} takes values in $\ker(T-\lambda_0)$ and acts trivially on $\operatorname{im}(T-\lambda_0) = \ker(T-\lambda_0)^{\perp}$ (recall that $\operatorname{im}(T-\lambda_0)$ is closed since $T-\lambda_0$ is Fredholm). The second identity implies that

$$0 = R_0(T-\lambda_0)w = w + R_{-1}w$$

for $w \in \ker(T-\lambda_0)$, i.e. R_{-1} acts as -1 times the identity on $\ker(T-\lambda_0)$. Putting both these things together we prove the proposition. □